PROPER HOLOMORPHIC EMBEDDINGS OF RIEMANN SURFACES WITH ARBITRARY TOPOLOGY INTO $\mathbb{C}^2$

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ABSTRACT. We prove that given an open Riemann surface $\mathcal{N}$, there exists an open domain $\mathcal{M} \subset \mathcal{N}$ homeomorphic to $\mathcal{N}$ which properly holomorphically embeds in $\mathbb{C}^2$. Furthermore, $\mathcal{M}$ can be chosen with hyperbolic conformal type. In particular, any open orientable surface M admits a complex structure $\mathcal{C}$ such that $(M, \mathcal{C})$ can be properly holomorphically embedded into $\mathbb{C}^2$.

1. INTRODUCTION

It is classically known that any open Riemann surface properly holomorphically embeds in $\mathbb{C}^3$ and immerses in $\mathbb{C}^2$ [**Re**, **Na1**, **Na2**, **Bi**]. Bell-Narasimhan's conjecture asserts that any open Riemann surface can be properly holomorphically embedded in $\mathbb{C}^2$ [**BN**, Conjecture 3.7, p. 20]. Although this old embeddability problem has generated vast literature, it still remains open.

The first existence results for discs and annuli can be found in [**St**] and [**La**, **Al**], respectively. More recently, it has been proved that any finitely connected planar domain without isolated boundary points properly holomorphically embeds into $\mathbb{C}^2$ [**GS**] (see also [**CG**]). Furthermore, any open orientable surface of finite topology admits a complex structure properly holomorphically embedding in $\mathbb{C}^2$ [**CF**]. In the last few years, this area has experimented a great grothw. Specially interesting are the works by Wold [**W1**, **W2**] and Forstnerič and Wold [**FW**] (see also [**Ma**]). These authors have shown that any bordered Riemann surface whose closure admits a (non-proper) holomorphic embedding into $\mathbb{C}^2$ actually properly holomorphically embeds into $\mathbb{C}^2$. (A bordered Riemann surface is the interior of a compact one-dimensional complex manifold with smooth boundary consisting of finitely many closed Jordan arcs.) In all these constructions, the (finite) topological type of the surface, and even its conformal structure, is not changed during the process.

The aim of this paper is to show that the topology of an open Riemann surface plays no role in this setting. We extend the above mentioned result by Černe and Forstnerič [**CF**] to the case of surfaces with arbitrary topology, proving the following topological version of Bell-Narasimhan's conjecture:

> **Main Theorem.** *Let $\mathcal{N}$ be an open Riemann surface.*
>
> *Then there exists an open domain $\mathcal{M} \subset \mathcal{N}$ homeomorphic to $\mathcal{N}$ carrying a proper holomorphic embedding $\mathcal{Y} : \mathcal{M} \to \mathbb{C}^2$.*
>
> *In particular, any open orientable surface M admits a complex structure $\mathcal{C}$ such that the Riemann surface $(M, \mathcal{C})$ properly holomorphically embeds in $\mathbb{C}^2$.*

The proper embedding $\mathcal{Y} : \mathcal{M} \to \mathbb{C}^2$ in Main Theorem is obtained as the limit of a sequence of holomorphic embeddings $\{Y_n : M_n \to \mathbb{C}^2\}_{n\in\mathbb{N}}$, where $\{M_n\}_{n\in\mathbb{N}}$ is a suitable expansive sequence of compact regions in $\mathcal{N}$ and $\mathcal{M} = \cup_{n\in\mathbb{N}} M_n$. The sequence is constructed by combining a bridge principle for holomorphic embeddings with Forstnerič and Wold's techniques.

It is worth mentioning that the open Riemann surface $\mathcal{M}$ in Main Theorem can be chosen of hyperbolic conformal type. Finally, let us point out that Main Theorem actually follows from a more general extension result for holomorphic embeddings into $\mathbb{C}^2$ (see Theorem 4.2).

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2. Preliminaries

As usual, we denote by $\|\cdot\|$ the Euclidean norm in $\mathbb{C}^n$, $n \in \mathbb{N}$, and for any compact topological space X and continuous map $f : X \to \mathbb{C}^n$ we denote by $\|f\| = \max\{\|f(p)\| \mid p \in X\}$ the maximum norm of f on X.

Non-compact Riemann surfaces without boundary are said to be *open*.

Remark 2.1. *Throughout this paper, $\mathcal{N}$ and ω will denote a fixed but arbitrary open Riemann surface and a complete smooth conformal metric on $\mathcal{N}$, respectively.*

For any $S \subset \mathcal{N}$, S° and $\overline{S}$ will denote the interior and the closure of S in $\mathcal{N}$, respectively.

Given a Riemann surface M contained in $\mathcal{N}$, we denote by ∂M the 1-dimensional topological manifold determined by its boundary points. Open connected subsets of $\mathcal{N}$ will be called *domains*, and those proper topological subspaces of $\mathcal{N}$ being Riemann surfaces with boundary are said to be *regions*.

A subset $S \subset \mathcal{N}$ is said to be *Runge* if $\mathcal{N} - S$ has no relatively compact components in $\mathcal{N}$, or equivalently, if the inclusion map $\jmath_S : S \hookrightarrow \mathcal{N}$ induces a group monomorphism $(\jmath_S)_* : \mathcal{H}_1(S,\mathbb{Z}) \to \mathcal{H}_1(\mathcal{N},\mathbb{Z})$. In this case we identify the groups $\mathcal{H}_1(S,\mathbb{Z})$ and $(\jmath_S)_*(\mathcal{H}_1(S,\mathbb{Z})) \subset \mathcal{H}_1(\mathcal{N},\mathbb{Z})$ via $(\jmath_S)_*$ and consider $\mathcal{H}_1(S,\mathbb{Z}) \subset \mathcal{H}_1(\mathcal{N},\mathbb{Z})$.

Two Runge subsets S_1, $S_2 \subset \mathcal{N}$ are said to be *isotopic* if $\mathcal{H}_1(S_1,\mathbb{Z}) = \mathcal{H}_1(S_2,\mathbb{Z})$. Two Runge subsets S_1, $S_2 \subset \mathcal{N}$ are said to be *homeomorphically isotopic* if there exists a homeomorphism $\sigma : S_1 \to S_2$ such that $\sigma_* = \mathrm{Id}_{\mathcal{H}_1(S_1,\mathbb{Z})}$, where σ_* is the induced group morphism on homology. In this case σ is said to be an *isotopical homeomorphism*. Two Runge domains with finite topology (or two Runge compact regions) in $\mathcal{N}$ are isotopic if and only if they are homeomorphically isotopic.

Let W be a Runge domain of finite topology in $\mathcal{N}$, and let S be a compact Runge subset in $\mathcal{N}$. W is said to be a *tubular neighborhood* of S if $S \subset W$ and S is isotopic to W. In addition, if $\overline{W}$ is a compact region isotopic to W then $\overline{W}$ is said to be a *compact tubular neighborhood* of S.

Definition 2.2 (Admissible set)**.** *A compact subset $S \subset \mathcal{N}$ is said to be admissible if and only if:*

- *$M_S := \overline{S^\circ}$ is a finite collection of pairwise disjoint compact regions in $\mathcal{N}$ with $\mathcal{C}^0$ boundary,*
- *$C_S := \overline{S - M_S}$ consists of a finite collection of pairwise disjoint analytical Jordan arcs,*
- *any component α of C_S with an endpoint $P \in M_S$ admits an analytical extension β in $\mathcal{N}$ such that the unique component of $\beta - \alpha$ with endpoint P lies in M_S, and*
- *S is Runge.*

For any subset $S \subset \mathcal{N}$, a function $f : S \to \mathbb{C}^n$, $n \in \mathbb{N}$, is said to be holomorphic if there exists a open set $U \subset \mathcal{N}$ containing S and a holomorphic function $h : U \to \mathbb{C}$ such that $h|_S = f$.

Definition 2.3. *Let $S \subset \mathcal{N}$ be an admissible set. A function $f : S \to \mathbb{C}^n$, $n \in \mathbb{N}$, is said to be* admissible *if $f|_{M_S}$ is holomorphic, and for any component α of C_S and any open analytical Jordan arc β in $\mathcal{N}$ containing α, f admits a smooth extension f_β to β satisfying that $f_\beta|_{U\cap\beta} = h|_{U\cap\beta}$, where $U \subset \mathcal{N}$ is an open domain containing M_S and $h : U \to \mathbb{C}^n$ is a holomorphic extension of f.*

Likewise, a complex 1-form θ of type $(1,0)$ on S is said to be *abmissible* if for any closed conformal disc (W,z) in $\mathcal{N}$ such that $W \cap S$ is admissible then $\theta|_{W\cap S} = g(z)dz$ for an admissible function $g : W \cap S \to \mathbb{C}$.

Given an admissible function $f : S \to \mathbb{C}^n$, we set df as the vectorial admissible 1-form given by $df|_{M_S} = d(f|_{M_S})$ and $df|_{\alpha\cap W} = (f \circ \alpha)'(x)dz|_{\alpha\cap W}$, where $(W, z = x+iy)$ is a conformal chart on $\mathcal{N}$ such that $\alpha \cap W = z^{-1}(\mathbb{R} \cap z(W))$.

If $f : S \to \mathbb{C}^n$ is admissible, then the $\mathcal{C}^1$-norm of f on S is given by

$$\|f\|_1 = \max_S(\|f\| + \|df/\omega\|).$$

3. Main Lemma

Set $\pi_1 : \mathbb{C}^2 \to \mathbb{C}$ the projection $\pi_1(z,w) = z$. We will need the following definition:

Definition 3.1 (**[W2, FW]**). *Let $M \subset \mathcal{N}$ be a Riemann surface possibly with boundary, and let $X : M \to \mathbb{C}^2$ be a proper holomorphic embedding. A point $p = (p_1, p_2)$ of the complex curve $\Sigma := X(M)$ is said to be* exposed (with respect to π_1) *if the complex line $\Lambda_p = \pi_1^{-1}(\pi_1(p)) = \{(p_1, w) \mid w \in \mathbb{C}\}$ intersects Σ only at p and this intersection is transverse, that is to say, $\Lambda_p \cap \Sigma = \{p\}$ and $T_p\Lambda_p \cap T_p\Sigma = \{0\}$.*

The proof of the following technical lemma is inspired by the ideas of Forstnerič and Wold [**W2**, **FW**]. Roughly speaking, Lemma 3.2 below asserts that an embedded bordered Riemman surface in $\mathbb{C}^2$ whose boundary lies outside an Euclidean ball can be perturbed near the boundary in such a way that the boundary of the arising surface lies outside a bigger ball in $\mathbb{C}^2$. The strength of this lemma is that embeddedness is preserved in this process.

For any $r > 0$ we denote by $\mathbb{B}(r) = \{z \in \mathbb{C}^2 \mid \|z\| < r\}$ and $\overline{\mathbb{B}}(r) = \{z \in \mathbb{C}^2 \mid \|z\| \leq r\}$.

Lemma 3.2. *Let M be a Runge compact region in $\mathcal{N}$, let $X : M \to \mathbb{C}^2$ be a holomorphic embedding and let $r > 0$ such that*

$$X(\partial M) \subset \mathbb{C}^2 - \overline{\mathbb{B}}(r). \tag{3.1}$$

Then, for any $\xi > 0$ and any $\hat{r} > r$, there exists a Runge compact region $\hat{M}$ on $\mathcal{N}$ and a holomorphic embedding $\hat{X} : \hat{M} \to \mathbb{C}^2$ satisfying that:

(L.1) *$\hat{M}$ is a compact tubular neighborhood of M,*
(L.2) *$\|\hat{X} - X\|_1 < \xi$ on M,*
(L.3) *$\hat{X}(\partial\hat{M}) \subset \mathbb{C}^2 - \overline{\mathbb{B}}(\hat{r})$, and*
(L.4) *$\hat{X}(\hat{M} - M^\circ) \subset \mathbb{C}^2 - \overline{\mathbb{B}}(r)$.*

Proof. Fix $\xi_0 \in]0, \xi[$ so that

$$X(\partial M) \subset \mathbb{C}^2 - \overline{\mathbb{B}}(r + \xi_0), \tag{3.2}$$

see (3.1). Take $\epsilon_0 > 0$ to be specified later.

We begin exposing boundary points as in [**FW**].

Since we are assuming that X holomorphically extends beyond M, there exists a Runge compact region N_1 on $\mathcal{N}$ and a holomorphic embedding $Y_1 : N_1 \to \mathbb{C}$ such that

(a.1) N_1 is a compact tubular neighborhood of M,
(a.2) $Y_1|_M = X$, and
(a.3) $Y_1(N_1 - M^\circ) \subset \mathbb{C}^2 - \overline{\mathbb{B}}(r + \xi_0)$.

Write $\partial M = \cup_{j=1}^m C_j$, where $\{C_j\}_{j=1}^m$ are the connected components of ∂M. Choose a point $a_j \in C_j$ and an analytic Jordan arc $\gamma_j \subset N_1^\circ - M^\circ$ with initial point a_j, otherwise disjoint from ∂M and such that the intersection of γ_j and C_j is transverse, $\forall j = 1, \ldots, m$. Take the arcs $\{\gamma_j\}_{j=1,\ldots,m}$ so that $M \cup (\cup_{j=1}^m \gamma_j)$ is admissible. Let b_j denote the other endpoint of γ_j, and let $U \subset N_1^\circ$ be a compact tubular neighborhood of M such that $b_j \notin U$, $U \cup (\cup_{j=1}^m \gamma_j)$ is admissible and $\tilde{\gamma}_j := \gamma_j \cap U$ is a Jordan arc with an endpoint at a_j, $j = 1, \ldots, m$.

On the other hand, consider pairwise disjoint smooth regular Jordan arcs $\{\lambda_j \mid j = 1, \ldots, m\}$ in $\mathbb{C}^2$ such that

(b.1) $Y_1(a_j)$ is an endpoint of λ_j, $Y_1(\tilde{\gamma}_j) \subset \lambda_j$ and $(\lambda_j - Y_1(\tilde{\gamma}_j)) \cap Y_1(U) = \emptyset$,
(b.2) $\lambda_j \subset \mathbb{C}^2 - \overline{\mathbb{B}}(r + \xi_0)$,
(b.3) the other endpoint p_j of λ_j satisfies $\Lambda_{p_j} \cap (Y_1(M) \cup (\cup_{i=1}^m \lambda_i)) = \{p_j\}$ and $T_{p_j}\Lambda_{p_j} \cap T_{p_j}^{\mathbb{C}}\lambda_j = \{0\}$, where $T_{p_j}^{\mathbb{C}}\lambda_j$ is the complexification of the real tangent line to λ_j at p_j, and
(b.4) $|\pi_1(p_j)| > r + \xi_0$,

for all $j \in \{1, \ldots, m\}$, see Figure 3.1. Notice that (b.3) is a generalization of Definition 3.1. Item (b.2) is possible since (a.3) holds.

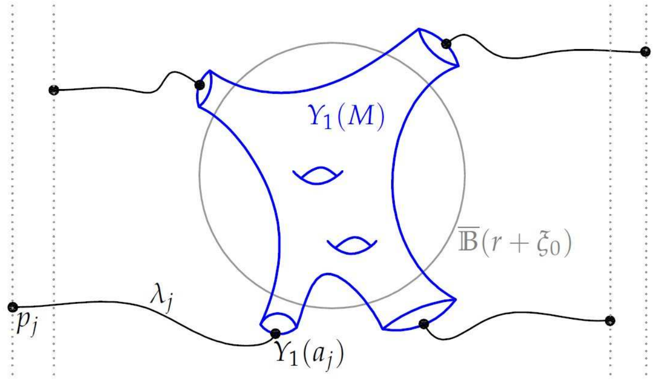

FIGURE 3.1. The arcs λ_j.

Consider an admissible embedding $\hat{Y}_1 : U \cup (\cup_{j=1}^m \gamma_j) \to \mathbb{C}^2$ such that

(c.1) $\hat{Y}_1|_U = Y_1$, and
(c.2) $\hat{Y}_1(\gamma_j) = \lambda_j$, $j = 1, \ldots, m$. In particular, $\hat{Y}_1(b_j) = p_j$.

By Mergelyan's Theorem (see for instance [**Fo**, Theorem 3.2]), we can find a Runge compact region N_2 and a holomorphic embedding $Y_2 : N_2 \to \mathbb{C}^2$ such that

(d.1) N_2 is a compact tubular neighborhood of U (hence, of M), with $\gamma_j \subset N_2 \subset N_1^\circ$ and $b_j \in \partial N_2$, $j = 1, \ldots, m$,
(d.2) $\|Y_2 - \hat{Y}_1\|_1 < \epsilon_0$ on $U \cup (\cup_{j=1}^m \gamma_j)$,
(d.3) $Y_2(N_2 - M^\circ) \subset \mathbb{C}^2 - \overline{\mathbb{B}}(r + \xi_0)$, and
(d.4) $Y_2(b_j) = \hat{Y}_1(b_j) = p_j$ is an exposed point for $Y_2(N_2)$.

Notice that (d.3) can be guaranteed from (a.3), (b.2), (c.1) and (c.2). Property (d.4) is possible thanks to (b.3) (see [**FW**, Theorem 4.2] for more details).

The second step in the proof of Lemma 3.2 consists of pushing $Y_2(\partial N_2)$ out of $\overline{\mathbb{B}}(\hat{r})$. Now we are inspired by [**W2**] and [**FW**, Theorem 5.1].

Write $\partial N_2 = \cup_{j=1}^m \Gamma_j$, where $\{\Gamma_j\}_{j=1}^m$ are the connected components of ∂N_2. Set

$$g : \mathbb{C}^2 \to \mathbb{C} \times \overline{\mathbb{C}}, \quad g(z, w) = \left(z, w + \sum_{j=1}^m \frac{\alpha_j}{z - \pi_1(Y_2(b_j))} \right), \tag{3.3}$$

where the coefficients $\alpha_j \in \mathbb{C} - \{0\}$ are chosen so that the following assertions hold.

(e.1) π_2 maps the curve $\mu_j := (g \circ Y_2)(\Gamma_j - \{b_j\}) \subset \mathbb{C}^2$ into an unbounded curve $\delta_j \subset \mathbb{C}$, and $\pi_2 : \mu_j \to \delta_j$ is a diffeomorphism near infinity, where $\pi_2 : \mathbb{C}^2 \to \mathbb{C}$ is given by $\pi_2(z, w) = w$.
(e.2) $\overline{\mathbb{D}}(\rho) \cup (\cup_{j=1}^m \delta_j)$ is Runge in $\mathbb{C}$ for any large enough $\rho \in \mathbb{R}$, where $\overline{\mathbb{D}}(\rho) = \{z \in \mathbb{C} \mid |z| \leq \rho\}$.
(e.3) $\|g \circ Y_2 - Y_2\|_1 < \epsilon_0$ on M.
(e.4) $(g \circ Y_2)((N_2 - \{b_j\}_{j=1,\ldots,m}) - M^\circ) \subset \mathbb{C}^2 - \overline{\mathbb{B}}(r + \xi_0)$.

This can be guaranteed by a careful choice of the argument of the complex number α_j, while $|\alpha_j|$ must be chosen as small as needed, $j = 1, \ldots, m$. To achieve properties (e.3) and (e.4), we argue as follows. First, fix pairwise disjoint small open discs $W_j \subset \mathcal{N}$, $j = 1, \ldots, m$, such that $b_j \in W_j$ and

$$|\pi_1(Y_2(W_j \cap N_2))| > r + \xi_0 \text{ for all } j, \tag{3.4}$$

see (b.4). Then choose $|\alpha_j|$, $j = 1, \ldots, m$, small enough so that $(g \circ Y_2)(N_2 - (M^\circ \cup_j W_j)) \subset \mathbb{C}^2 - \overline{\mathbb{B}}(r + \xi_0)$ (see (d.3)) and $\|g \circ Y_2 - Y_2\|_1 < \epsilon_0$ on M. As $\pi_1 \circ g = \pi_1$, then (3.4) gives that $(g \circ Y_2)(N_2 \cap W_j) \subset \mathbb{C}^2 - \overline{\mathbb{B}}(r + \xi_0)$ as well.

Label $W := N_2 - \{b_j\}_{j=1,\ldots,m}$, set $Z : W \to \mathbb{C}^2$, $Z := g \circ Y_2|_W$, and note that Z is a well defined holomorphic embedding thanks to (d.4). Furthermore, Z has the following property: there exists a compact polynomially convex $K_0 \subset Z(W)$ in $\mathbb{C}^2$ such that $K := K_0 \cup \overline{\mathbb{B}}(r + \xi_0)$ is polynomially convex and $Z(M) \subset K \subset \mathbb{C}^2 - Z(\partial W)$ (see (e.4) and the proof of Theorem 5.1 in **[FW]**). Moreover there exists a holomorphic automorphism ϕ of $\mathbb{C}^2$ such that

(f.1) $Z(M) \cup \overline{\mathbb{B}}(r + \xi_0) \subset K \subset \mathbb{C}^2 - Z(\partial W)$, notice that $\partial W = \partial N_2 - \{b_j\}_{j=1,\ldots,m}$,
(f.2) $\|\phi - \mathrm{Id}_{\mathbb{C}^2}\| < \epsilon_0$ on K, and $\|\phi \circ Z - Z\|_1 < \epsilon_0$ on M, and
(f.3) $(\phi \circ Z)(\partial W) \subset \mathbb{C}^2 - \overline{\mathbb{B}}(\hat{r})$.

Such ϕ is constructed in **[W1]** from (e.1) and (e.2), see also the proof of Theorem 5.1 in **[FW]**.

Define $\hat{X} : W \to \mathbb{C}^2$, $\hat{X} := \phi \circ Z$, and let us check that $\hat{X}$ *almost* satisfies the conclusion of Lemma 3.2.

- W° is an open tubular neighborhood of M. See (d.1) and the definitions of U and W.
- $\|\hat{X} - X\|_1 < \xi$ on M. Indeed, use (a.2), (d.2), (e.3), (f.1) and (f.2) and assume that ϵ_0 was chosen small enough from the beginning.
- $\hat{X}(\partial W) \subset \mathbb{C}^2 - \overline{\mathbb{B}}(\hat{r})$. See (f.3).
- $\hat{X}(W - M^\circ) \subset \mathbb{C}^2 - \overline{\mathbb{B}}(r)$. Indeed, if ϵ_0 is chosen small enough from the begining, then taking into account (f.1), (f.2) and that ϕ is bijective, we conclude that $\phi(\mathbb{C}^2 - \overline{\mathbb{B}}(r + \xi_0)) \subset \mathbb{C}^2 - \phi(\overline{\mathbb{B}}(r))$. Then use (e.4).

Taking into account these properties of $\hat{X}$, we finish by setting $\hat{M}$ as a suitable shrinking of W satisfying (L.1). The proof is done. □

4. Main Theorem

We will need the following

Definition 4.1. *Let K be a compact subset of $\mathcal{N}$, let $f : K \to \mathbb{C}^2$ be a topological embedding and let $n \in \mathbb{N}$. We define*

$$\Psi(K, f, n) := \frac{1}{2n^2} \inf\left\{ \|f(p) - f(q)\| \;\middle|\; p, q \in K,\ \mathsf{d}(p,q) > \frac{1}{n} \right\},$$

where $\mathsf{d}(\cdot,\cdot)$ means distance in the Riemannian surface $(\mathcal{N}, \omega)$, see Remark 2.1. Notice that $\Psi(K, f, n) > 0$.

Now we can state and prove the main result of this paper.

Theorem 4.2. *Let N be a Runge compact region on $\mathcal{N}$ and let $Y : N \to \mathbb{C}^2$ be a holomorphic embbeding. Assume that*

$$Y(\partial N) \subset \mathbb{C}^2 - \overline{\mathbb{B}}(s) \tag{4.1}$$

for a positive s.

Then, for any $\epsilon > 0$, there exist an open domain $\mathcal{M} \subset \mathcal{N}$ and a proper holomorphic embedding $\mathcal{X} : \mathcal{M} \to \mathbb{C}^2$ satisfying

(T.1) *$N \subset \mathcal{M}$, $\mathcal{M}$ is Runge and isotopic to $\mathcal{N}$,*
(T.2) *$\|\mathcal{X} - Y\|_1 < \epsilon$ on N, and*
(T.3) *$\mathcal{X}(\mathcal{M} - N^\circ) \subset \mathbb{C}^2 - \overline{\mathbb{B}}(s)$.*

Proof. Consider an exhaustion $\{M_j\}_{j\in\mathbb{N}}$ of $\mathcal{N}$ by Runge compact regions so that $M_1 = N$, and $M_{j-1} \subset M_j^\circ$ and the Euler characteristic $\chi(M_j - M_{j-1}^\circ) \in \{-1, 0\}$ for all $j \geq 2$ (if $\mathcal{N}$ has finite topology then $\chi(M_j - M_{j-1}^\circ) = 0$ for any large enough j).

Since $Y(\partial N)$ is compact, equation (4.1) guarantees the existence of $s_0 > s$ such that

$$Y(\partial N) \subset \mathbb{C}^2 - \overline{\mathbb{B}}(s_0). \tag{4.2}$$

Take $\epsilon_0 > 0$ with

$$\epsilon_0 < \min\{\epsilon, s_0 - s\}, \tag{4.3}$$

to be specified later.

Claim 4.3. *There exists a sequence* $\{\Xi_j\}_{j\in\mathbb{N}} := \{(N_j, \sigma_j, Y_j, \epsilon_j)\}_{j\in\mathbb{N}}$, *where*

- N_j *is a Runge compact region on* $\mathcal{N}$ *isotopic to* M_j,
- $\sigma_j : N_j \to M_j$ *is an isotopic homeomorphism,*
- $Y_j : N_j \to \mathbb{C}^2$ *is a holomorphic embedding, and*
- $\epsilon_j > 0$, $j \in \mathbb{N}$,

such that

(A$_j$) $N_{j-1} \subset N_j^\circ$ *and* $\sigma_j|_{N_{j-1}} = \sigma_{j-1}$,
(B$_j$) $\epsilon_j < \min\{\epsilon_0/2^j\,,\, \Psi(N_{j-1}, Y_{j-1}, j)\,,\, \epsilon_{j-1}\,,\, \varrho_{j-1}\}$, *where*

$$\varrho_{j-1} := \frac{1}{2^j}\min\left\{\min_{N_k}\|\frac{dY_k}{\omega}\| \mid k = 1,\ldots,j-1\right\} > 0,$$

(C$_j$) $\|Y_j - Y_{j-1}\|_1 < \epsilon_j$ *on* N_{j-1},
(D$_j$) $Y_j(\partial N_j) \subset \mathbb{C}^2 - \overline{\mathbb{B}}(s_0 + j - 1)$, *and*
(E$_j$) $Y_j(N_j - N_{j-1}^\circ) \subset \mathbb{C}^2 - \overline{\mathbb{B}}(s_0 + j - 2)$.

Proof. The sequence is constructed inductively. Set $\Xi_1 := (N, \mathrm{Id}|_N, Y, \epsilon_1)$, where $\epsilon_1 < \epsilon_0/4$. Equation (4.2) gives property (D$_1$) whereas properties (A$_1$), (B$_1$), (C$_1$) and (E$_1$) do not make sense.

To prove the inductive step, assume that $\Xi_1, \ldots, \Xi_{j-1}$ are already constructed satisfying the required properties and let us construct Ξ_j, $j \geq 2$.

We need to distinguish two cases depending on $\chi(M_j - M_{j-1}^\circ)$.

• **Case 1.** Assume $\chi(M_j - M_{j-1}^\circ) = 0$. Apply Lemma 3.2 to the data

$$M = N_{j-1}, \quad X = Y_{j-1}, \quad r = s_0 + j - 2, \quad \xi = \epsilon_j \quad \text{and} \quad \hat{r} = s_0 + j - 1,$$

where ϵ_j is any positive satisfying (B$_j$). Observe that the lemma can be applied thanks to property (D$_{j-1}$). Then we set $\Xi_j := (N_j = \hat{M}, \sigma_j, Y_j = \hat{X}, \epsilon_j)$, where $\hat{M}$ and $\hat{X}$ are the data arising from the lemma and $\sigma_j : N_j \to M_j$ is any homeomorphism with $\sigma_j|_{N_{j-1}} = \sigma_{j-1}$. Properties (A$_j$), (C$_j$), (D$_j$) and (E$_j$) directly follow from (L.1), (L.2), (L.3) and (L.4) of Lemma 3.2, respectively.

• **Case 2.** Assume $\chi(M_j - M_{j-1}^\circ) = -1$. First of all, fix $\epsilon_j > 0$ satisfying (B$_j$). Take a Runge compact region R_1 and a holomorphic embedding $Z_1 : R_1 \to \mathbb{C}^2$ such that

(a.1) R_1 is a compact tubular neighborhood of N_{j-1},
(a.2) $Z_1|_{N_{j-1}} = Y_{j-1}$, and
(a.3) $Z_1(R_1 - N_{j-1}^\circ) \subset \mathbb{C}^2 - \overline{\mathbb{B}}(s_0 + j - 2)$.

(Recall that Y_{j-1} extends holomorphically beyond N_{j-1} in $\mathcal{N}$.) Consider a smooth Jordan curve $\hat{\alpha} \in \mathcal{H}_1(M_j, \mathbb{Z}) - \mathcal{H}_1(M_{j-1}, \mathbb{Z})$ contained in M_j° and intersecting $M_j - M_{j-1}^\circ$ in a Jordan arc α with endpoints a, b in ∂M_{j-1} and otherwise disjoint from M_{j-1}. Since M_{j-1} and M_j are Runge and $\chi(M_j - M_{j-1}^\circ) = -1$, then $\mathcal{H}_1(M_j, \mathbb{Z}) = \mathcal{H}_1(M_{j-1} \cup \alpha, \mathbb{Z})$ and $M_{j-1} \cup \alpha$ is Runge as well. Take an analytic Jordan arc $\gamma \subset \mathcal{N} - N_{j-1}^\circ$ with endpoints $\sigma_{j-1}^{-1}(a)$, $\sigma_{j-1}^{-1}(b)$ in ∂N_{j-1}, otherwise disjoint from N_{j-1}, transversally intersecting ∂N_{j-1} and such that $N_{j-1} \cup \gamma$ is admissible. Take also an isotopic homeomorphism $\varsigma : N_{j-1} \cup \gamma \to M_{j-1} \cup \alpha$ so that $\varsigma|_{N_{j-1}} = \sigma_{j-1}$ and $\varsigma(\gamma) = \alpha$.

On the other hand, consider in $\mathbb{C}^2$ a smooth regular Jordan arc λ agreeing with $Z_1(\gamma \cap R_1)$ near the endpoints $Z_1(\sigma_{j-1}^{-1}(a))$ and $Z_1(\sigma_{j-1}^{-1}(b))$, and such that

(b.1) $(\lambda - Z_1(\gamma \cap R_1)) \cap Z_1(N_{j-1}) = \emptyset$, and
(b.2) $\lambda \subset \mathbb{C}^2 - \overline{\mathbb{B}}(s_0 + j - 2)$.

This choice of λ is possible thanks to property (a.3). Consider the admissible embedding $\hat{Z}_1 : N_{j-1} \cup \gamma \to \mathbb{C}^2$ given by $\hat{Z}_1|_{N_{j-1}} = Z_1$ and $\hat{Z}_1(\gamma) = \lambda$. Mergelyan's Theorem provides Runge a compact region R_2 and a holomorphic embedding $Z_2 : R_2 \to \mathbb{C}^2$ satisfying that

(c.1) R_2 is a compact tubular neighborhoods of $N_{j-1} \cup \gamma$,
(c.2) $\|Z_2 - \hat{Z}_1\|_1 < \epsilon_j/2$ on $N_{j-1} \cup \gamma$, and
(c.3) $Z_2(R_2 - N_{j-1}^\circ) \subset \mathbb{C}^2 - \overline{\mathbb{B}}(s_0 + j - 2)$.

Since $Z_2(R_2 - N_{j-1}^\circ)$ is compact, (c.3) implies the existence of $\varepsilon \in]0, \epsilon_j/2[$ small enough so that

$$Z_2(R_2 - N_{j-1}^\circ) \subset \mathbb{C}^2 - \overline{\mathbb{B}}(s_0 + j - 2 + \varepsilon). \tag{4.4}$$

Set $\Xi_j := (N_j = \hat{M}, \sigma_j, Y_j = \hat{X}, \epsilon_j)$, where $\hat{M}$ and $\hat{X}$ are the data arising from Lemma 3.2 applied to the data

$$M = R_2, \quad X = Z_2, \quad r = s_0 + j - 2 + \varepsilon, \quad \xi = \varepsilon \quad \text{and} \quad \hat{r} = s_0 + j - 1,$$

where $\sigma_j : N_j \to M_j$ is any homeomorphism with $\sigma_j|_{N_{j-1} \cup \gamma} = \varsigma$. Observe that the lemma can be applied thanks to (c.3). Property (A$_j$) follows from (a.1), (c.1) and Lemma 3.2-(L.1). Property (C$_j$) is implied by (a.2), (c.2) and Lemma 3.2-(L.2). Item (L.3) in Lemma 3.2 gives (D$_j$). Finally, to check (E$_j$) consider a point $p \in N_j - N_{j-1}^\circ$ and let us distinguish cases. If $p \in N_j - R_2^\circ$ then Lemma 3.2-(L.4) gives $Y_j(p) \in \mathbb{C}^2 - \overline{\mathbb{B}}(s_0 + j - 2)$ and we are done. Otherwise $p \in R_2 - N_{j-1}^\circ$, and in this case Lemma 3.2-(L.2) and equation (4.4) guarantee that $Y_j(p) \in \mathbb{C}^2 - \overline{\mathbb{B}}(s_0 + j - 2)$ as well.

This concludes the construction of the sequence $\{\Xi_j\}_{j \in \mathbb{N}}$ satisfying the desired properties. □

Set $\mathcal{M} := \cup_{j \in \mathbb{N}} N_j$ and $\sigma : \mathcal{M} \to \mathcal{N}$, $\sigma|_{N_j} = \sigma_j$. Since $\{M_j\}_{j \in \mathbb{N}}$ is an exhaustion of $\mathcal{N}$ by Runge compact regions and σ_j is an isotopic homeomorphism for all j, then σ is an isotopic homeomorphism as well and statement (T.1) holds.

Properties (B$_j$) and (C$_j$), $j \in \mathbb{N}$, imply that the sequence of holomorphic maps $\{Y_j\}_{j \in \mathbb{N}}$ uniformly converges on compact subsets of $\mathcal{M}$ to a holomorphic map $\mathcal{X} : \mathcal{M} \to \mathbb{C}^2$ satisfying

$$\|\mathcal{X} - Y\|_1 < \epsilon_0 \quad \text{on } N. \tag{4.5}$$

(Recall that $Y_1 = Y$ and $N_1 = N$). This implies (T.2) (see equation (4.3)).

Let us check (T.3). Take $p \in \mathcal{M} - N^\circ$. Then, there exists $j \geq 2$ such that $p \in N_j - N_{j-1}^\circ$ and, by properties (C$_j$) and (E$_j$), $\mathcal{X}(p) \in \mathbb{C}^2 - \overline{\mathbb{B}}(s_0 + j - 2 - \epsilon_0) \subset \mathbb{C}^2 - \overline{\mathbb{B}}(s)$, see (4.3).

To check that $\mathcal{X}$ is injective we have to work a little further. Take $p, q \in \mathcal{M}$, $p \neq q$, and let us prove that $\mathcal{X}(p) \neq \mathcal{X}(q)$. Indeed, consider a large enough $j_0 \in \mathbb{N}$ so that $\{p, q\} \subset N_j$ and $\mathrm{d}(p, q) > 1/j$, $\forall j \geq j_0$. Then, for any $j > j_0$, from properties (B$_j$) and (C$_j$), one has

$$\begin{aligned} \|Y_{j-1}(p) - Y_{j-1}(q)\| &\leq \|Y_{j-1}(p) - Y_j(p)\| + \|Y_j(p) - Y_j(q)\| + \|Y_j(q) - Y_{j-1}(q)\| \\ &< 2\epsilon_j + \|Y_j(p) - Y_j(q)\| \\ &\leq \frac{1}{j^2}\|Y_{j-1}(p) - Y_{j-1}(q)\| + \|Y_j(p) - Y_j(q)\|, \end{aligned}$$

see Definition 4.1. Therefore, $\|Y_j(p) - Y_j(q)\| \geq (1 - 1/j^2)\|Y_{j-1}(p) - Y_{j-1}(q)\|$, $\forall j > j_0$, and so

$$\|Y_{j_0+i}(p) - Y_{j_0+i}(q)\| \geq \|Y_{j_0}(p) - Y_{j_0}(q)\| \cdot \prod_{j=j_0+1}^{j_0+i} \left(1 - \frac{1}{j^2}\right), \quad \forall i \in \mathbb{N}.$$

Taking limits in the above inequality as $i \to \infty$ we obtain that $\|\mathcal{X}(p) - \mathcal{X}(q)\| \geq \frac{1}{2}\|Y_{j_0}(p) - Y_{j_0}(q)\| > 0$ (recall that Y_{j_0} is an embedding) and we are done.

Let us check that $\mathcal{X} : \mathcal{M} \to \mathbb{C}^2$ is proper. Consider a compact subset $K \subset \mathbb{C}^2$. It suffices to prove that $\mathcal{X}^{-1}(K)$ is compact in $\mathcal{M}$. Take $j_0 \in \mathbb{N}$ large enough so that $K \subset \overline{\mathbb{B}}(s_0 + j_0 - 2 - \epsilon_0)$. On the other hand, properties (B$_j$) and (E$_j$) give that $\mathcal{X}(N_j - N_{j-1}^\circ) \subset \mathbb{C}^2 - \overline{\mathbb{B}}(s_0 + j_0 - 2 - \epsilon_0)$ for any $j \geq j_0$. Hence $\mathcal{X}^{-1}(K) \subset N_{j_0-1}$ which is compact in $\mathcal{M}$, and we are done.

Finally, let us check that $\mathcal{X}$ is an immersion, hence an embedding. Take $p \in \mathcal{M}$ and $j_0 \in \mathbb{N}$ such that $p \in N_j \; \forall j \geq j_0$. Then

$$\begin{aligned} \|d\mathcal{X}/\omega\|(p) &\geq \|dY_{j_0}/\omega\|(p) - \sum_{j>j_0} \|Y_j - Y_{j-1}\|_1 \geq \|dY_{j_0}/\omega\|(p) - \sum_{j>j_0} \epsilon_j \\ &\geq \|dY_{j_0}/\omega\|(p) - \sum_{j>j_0} \varrho_{j-1} \geq \|dY_{j_0}/\omega\|(p)\big(1 - \sum_{j>j_0} \frac{1}{2^j}\big) \geq \frac{1}{2}\|dY_{j_0}/\omega\|(p) > 0, \end{aligned}$$

where we have used (B$_j$), $j > j_0$. The proof of Theorem 4.2 is done. □

Main Theorem in the introduction easily follows from Theorem 4.2. Indeed, let $\mathcal{N}$ be an open Riemann surface, let N be a conformal compact disc on $\mathcal{N}$ and let $Y : N \to \mathbb{C}^2$ be a holomorphic embedding with $Y(\partial N) \subset \mathbb{C}^2 - \overline{\mathbb{B}}(1)$. Then Theorem 4.2 provides an open domain $\mathcal{M} \subset \mathcal{N}$ homeomorphic to $\mathcal{N}$ and a proper holomorphic embedding $\mathcal{X} : \mathcal{M} \to \mathbb{C}^2$. Furthermore, if we substitute $\mathcal{N}$ for any hiperbolic isotopic subdomain of $\mathcal{N}$, the arising domain $\mathcal{M}$ is hyperbolic as well.

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